Pergamon

Computers Math. Applic. Vol. 27, No. 12, pp. 1–6, 1994

0898-1221/94 $7.00 + 0.00

0898-1221(94)E0076-V

The Error Analysis for Degree Reduction of Bézier Curves

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(Received and accepted February 1994)

Abstract—The error analysis of Farin's and Forrest's algorithms for generating an approximation of degree $n-1$ to an n^{th} degree Bézier curve is presented. Algorithms are based on observations of the geometric properties of the Bézier curve which allow the development of detailed error analysis. By combining subdivision with a degree reduction algorithm, a piecewise approximation can be generated, which is within some preset error tolerance of the original curve. The number of subdivisions required can be determined *a priori* and a piecewise approximation of degree m can be generated by iterating the scheme.

1. INTRODUCTION

In general, degree reduction of Bézier curves address the following problem.

PROBLEM 1. *Let $\{b_i\}_{i=0}^n \subset \mathrm{R}^s$ be a given set of (control) points which define the Bézier curve*

$$b^n(t) = \sum_{i=0}^{n} b_i B_i^n(t), \qquad 0 \le t \le 1$$

in terms of Bernstein polynomials $B_i^n(t) = \binom{n}{i}(1-t)^{n-i}t^i$ of degree n. Then find another point set $\{q_i\}_{i=0}^m \subset \mathrm{R}^s$ defining the (approximative) Bézier curve

$$q^m(t) = \sum_{i=0}^{m} q_i B_i^m(t), \qquad 0 \le t \le 1$$

of lower degree $m < n$ so that a suitable distance function $d(b^n, q^m)$ between b^n and q^m is minimized on the interval $[0,1]$.

One of the main uses of a degree reduction algorithm is to generate a piecewise linear approximation to a prescribed curve or surface. These piecewise linear approximations are important because of their use in rendering, curve-curve, curve-surface and surface-surface intersection calculations. To generate a piecewise linear approximation to within a specified error tolerance, one must combine a subdivision algorithm with degree reduction. For Bézier curves and surfaces such algorithms are easy to implement. Subdivision/degree reduction in two and three variables has been investigated by Petersen [1] in the context of intersection problems.

This work was partially supported by KOSEF under Grant 91-08-00-01.

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The degree reduction is relevant also to other application areas such as CAGD (Computer Aided Geometric Design). The application of the algorithm is transferring data from one geometric modeling system to another. Often a curve of high degree must be approximated by a number of curves of lower degree, due to the limitation on the maximum polynomial degree that certain systems can store and work with. Such situation often arise, in practice, and some work in this area has been done. See, for instance, Dannenberg and Nowacki [2] or Hoscheck [3].

The necessity to determine the degree reduced curve by approximation is manifest since in general degree reduction is not exactly possible in contrast to the reversed question of degree elevation. Doing so, the degree reduction can be accomplished in number of ways. Hersch [4] developed a degree reduction algorithm for font conversion without error analysis. Sederberg and Kakimoto [5] consider the problem for approximating rational curves using polynomial curves. Watkins and Worsey [6] and Eck [7] developed an algebraic method for approximation using Chebycheff polynomials.

Now, the approximation q^m depends extremely on the chosen distance or error function $d(b^n, q^m)$ to be minimized. Here, the most appropriate metric in geometrical terms would be the *Hausdorff distance*; see Degen [8] for a detailed discussion. It measures a special kind of geometrical distance between the two compact point sets $\{b^n(t) : t \in [0,1]\}$ and $\{q^m(t) : t \in [0,1]\}$ by ignoring the respective parameterizations. The according approximation process is explained in Degen [8] and Eisele [9] where nonlinear theory is used to determine the curve q^m numerically.

On the other hand, we are always interested in explicit formulas for the approximation q^m. Therefore, we will use the much simpler distance function

$$d(b^n, q^m) = \max\{\|b^n(t) - q^m(t)\| : t \in [0,1]\}, \tag{1.1}$$

which measures the maximal distance with respect to the parameterization. An immediate consequence of this error metric $d(b^n, q^m)$ is that the geometry of b^n is approximated as well as its parameterization.

The main advantage of using (1.1) is the decomposition of original Problem 1 into s subproblems. Each subproblem consists of minimization the componentwise error function

$$d\left(b_j^n, q_j^m\right) = \max\left\{|b_j^n(t) - q_j^m(t)| : t \in [0,1]\right\}, \qquad j = 1, \dots, s$$

if we introduce the notation $f(t) = (f_1(t), \dots, f_s(t))^\top$. Then the minimal distance (1.1) is determined by $d(b^n, q^m) = (\sum_{j=1}^s d(b_j^n, q_j^m)^2)^{1/2}$.

Hence, it is sufficient from now on to investigate the single-valued or functional case only. Furthermore, we will restrict ourselves to the very special case $m = n-1$ since explicit solutions are only known for that exceptional case. This is carried out in Section 2 where the following reformulated problem is solved.

PROBLEM 2. *Let $\{b_i\}_{i=0}^n$ be a given set of real coefficients which define the Bézier function*

$$b^n(t) = \sum_{i=0}^n b_i B_i^n(t), \qquad 0 \le t \le 1.$$

Then find another real coefficients $\{q_i\}_{i=0}^{n-1}$ defining the approximation

$$q^{n-1}(t) = \sum_{i=0}^{n-1} q_i B_i^m(t), \qquad 0 \le t \le 1$$

by minimizing the uniform error function $d(b^n, q^{n-1}) = \max\left\{|b^n(t) - q^{n-1}(t)|\right\}$ on interval $[0,1]$.

In the second section of the paper, the error analysis for Farin's [10] and Forrest's [11] methods are presented. The authors aim to develop a geometric method for generating lower degree approximations which, in the uniform norm, are within some preset error tolerance of the prescribed Bézier curve. This means that the scheme for approximation must be combined with a subdivision algorithm. This question is addressed in the third section. The techniques used in the approximation algorithm admit a detailed error analysis for the method. These results are presented and used to determine, *a priori*, the number of subdivisions that are needed for the approximation to be within the error tolerance that is imposed. The original curve is subdivided and each segment of it is approximated separately.

2. ERROR ANALYSIS FOR DEGREE REDUCTION

If $b^n(t)$ would give the $n+1$ coefficients of the degree elevated function $q^{n-1}(t)$ are determined by (cf. Farin [12]):

$$b_i = \frac{i}{n} q_{i-1} + \left(1 - \frac{i}{n}\right) q_i, \qquad i = 0, \dots, n. \tag{2.1}$$

Now, the equations (2.1) can be used to derive two different recursive extrapolation formulas for the generation of the $\{q_i\}$ from the $\{b_i\}$:

$$q_0^l = b_0, \quad q_i^l = \frac{n}{n-i} b_i - \frac{i}{n-i} q_{i-1}^l, \qquad i = 1, \dots, n-1 \tag{2.2}$$

or

$$q_{n-1}^r = b_n, \quad q_{i-1}^r = \frac{n}{i} b_i - \frac{n-i}{i} q_i^r, \qquad i = n-1, \dots, 1. \tag{2.3}$$

Both formulas only represent approximations and immediately observes that (2.2) produces reasonable approximations near to b_0 and the (2.3) behaves decently near b_n. Therefore, Farin [10] proposed the method by taking a weighted average of the form

$$q_i = \left(1 - \frac{i}{n-1}\right) q_i^l + \frac{i}{n-1} q_i^r, \qquad i = 0, \dots, n-1.$$

For the error analysis, we obtain the following lemma from equation (2.2) and (2.3).

LEMMA 1. *The coefficients q_i^l and q_i^r are rewritten as*

$$q_i^l = \frac{(-1)^i}{\binom{n-1}{i}} \sum_{j=0}^{i} (-1)^j \binom{n}{j} b_j \tag{2.4}$$

and

$$q_i^r = \frac{(-1)^{i+1}}{\binom{n-1}{i}} \sum_{j=i+1}^{n} (-1)^j \binom{n}{j} b_j. \tag{2.5}$$

In the Farin's method, the approximation is defined as

$$q^{n-1}(t) = \sum_{i=0}^{n-1} \left(\frac{n-1-i}{n-1} q_i^l + \frac{i}{n-1} q_i^r \right) B_i^{n-1}(t).$$

We obtain at first with help of Lemma 1 that

$$q^{n-1}(t) = \sum_{i=0}^{n-1} \frac{(-1)^i}{\binom{n-1}{i}} \left(V_i - \frac{i}{n-1} V_n \right) B_i^{n-1}(t), \tag{2.6}$$

where $V_r = \sum_{j=0}^{r} (-1)^j \binom{r}{j} b_j$.

If we artificially degree elevate the q^{n-1}, then

$$q^{n-1}(t) = \sum_{i=0}^{n} \left(\frac{n-i}{n} q_i + \frac{i}{n} q_{i-1} \right) B_i^n(t), \tag{2.7}$$

where the coefficients q_{-1} and q_n are negligible.

Applying this to (2.6),

$$\begin{aligned} q^{n-1}(t) &= \sum_{i=0}^{n} \left(b_i - \frac{(-1)^i V_n}{(n-1)\binom{n}{i}} \right) B_i^n(t) \\ &= b^n(t) - \frac{V_n}{n-1} \sum_{i=0}^{n} \frac{(-1)^i}{\binom{n}{i}} B_i^n(t). \end{aligned}$$

Thus, we have

$$b^n(t) - q^{n-1}(t) = \frac{V_n}{n-1} \sum_{i=0}^{n} \frac{(-1)^i}{\binom{n}{i}} B_i^n(t)$$

and

$$\max_{0 \le t \le 1} \left| b^n(t) - q^{n-1}(t) \right| \le \frac{1}{n-1} |V_n|.$$

On the other hand, Forrest [11] proposed to combine both formulas by taking the left half of the coefficients from (2.2) and right half of the coefficients from (2.3) as

$$\begin{aligned} q_i &= q_i^l, \qquad i = 0, 1, \dots, \\ q_i &= q_i^r, \qquad i = n-1, n-2, \dots . \end{aligned}$$

For odd n, the appearing midpoint is defined by $q_i = 1/2(q_i^l + q_i^r)$ with $i = (n-1)/2$.

With help of Lemma 1, the coefficients of the artificially degree elevated function q^{n-1} in (2.7) are obtained as

$$\frac{n-i}{n} q_i + \frac{i}{n} q_{i-1} = b_i - \frac{(-1)^i}{\binom{n}{i}} V_n, \qquad i = \frac{n}{2} \ (n : \text{even}) \quad \text{and}$$

$$\frac{n-i}{n} q_i + \frac{i}{n} q_{i-1} = b_i - \frac{(-1)^i}{2\binom{n}{i}} V_n, \qquad i = \frac{n-1}{2},\ \frac{n+1}{2} \ (n : \text{odd}),$$

otherwise

$$\frac{n-i}{n} q_i + \frac{i}{n} q_{i-1} = b_i.$$

Thus, we have for even n,

$$q^{n-1}(t) = b^n(t) - \frac{(-1)^{n/2}}{\binom{n}{\frac{n}{2}}} V_n B_{n/2}^n(t)$$

and, for odd n,

$$q^{n-1}(t) = b^n(t) - \frac{(-1)^{(n-1)/2}}{2\binom{n}{\frac{n-1}{2}}} V_n \left\{ B_{(n-1)/2}^n(t) - B_{(n+1)/2}^n(t) \right\}.$$

Since the Bernstein polynomial B_i^n has only one maximum and attains it at $t = i/n$,

$$\max_{0 \le t \le 1} \left| b^n(t) - q^{n-1}(t) \right| \le \begin{cases} \left(\frac{1}{2}\right)^n |V_n|, & n \text{ is even,} \\ \frac{1}{2\sqrt{n}} \left(\frac{1}{4} - \frac{1}{4n}\right)^{(n-1)/2} |V_n|, & n \text{ is odd.} \end{cases}$$

Now, summarizing the results in this section, we reach the following theorem.

THEOREM 2. *The approximation error of the Farin's and Forrest's degree reduction method is given by*

$$d\left(b^n, q^{n-1}\right) \le \alpha_n |V_n|, \tag{2.8}$$

where $\alpha_n = 1/(n-1)$ *for the Farin's method and*

$$\alpha_n = \begin{cases} \left(\frac{1}{2}\right)^n, & n \text{ is even},\\ \frac{1}{2\sqrt{n}}\left(\frac{1}{4} - \frac{1}{4n}\right)^{(n-1)/2}, & n \text{ is odd}, \end{cases}$$

in the Forrest's method.

The replacement curve is to approximate the original curve in the sense that the two should lie within some provided tolerance (ϵ) of each other. We thus check the error function $d(b^n, q^{n-1})$

3. SUBDIVISION/DEGREE REDUCTION

In this section, we give an error analysis of the subdivision and degree reduction process. Our results show when and how often one should subdivide and/or attempt degree reduction. In particular, it is shown that for a curve b^n of degree n, it is possible to determine *a priori* how many times one must subdivide b^n before it can be approximated to within a given tolerance by curves of degree one less. Thus, it is not necessary to attempt approximation after each subdivision.

A Bézier curve b^n is usually defined over the interval $[0,1]$, but it can also be defined over any interval $[0,c]$. The part of the curve that corresponds to $[0,c]$ can also be defined by a Bézier polygon. Finding this Bézier polygon is referred to as subdivision of the Bézier curve. Recursive subdivision of Bézier curves is based on de Casteljau's theorem (see [12]).

Let us denote the Bézier polygon corresponding to the interval $[0,c]$ by $c_0, \dots, c_n$ - it defines a Bézier curve c^n (which is part of the same curve as b^n is, of course). We have the following subdivision formula for Bézier curves:

$$c_j = b_0^j(c),$$

where $b_0^j(c)$ is recursively defined by

$$b_i^j(c) = (1-c)b_i^{j-1}(c) + c b_{i+1}^{j-1}(c), \qquad \begin{cases} j = 1, \dots, n,\\ i = 0, \dots, n-j, \end{cases} \tag{3.1}$$

and $b_i^0(c) = b_i$. The control vertices corresponding to $[c,1]$ are given by the b_{n-j}^j. From the formula (3.1), we obtain the following lemma,

LEMMA 3. *Another formula for* c_j *is*

$$c_j = \begin{cases} \sum\limits_{k=0}^{j} B_k^j(c) b_k, & \text{for } [0,c],\\ \sum\limits_{k=0}^{j} B_k^j(c) b_{n-j+k}, & \text{for } [c,1]. \end{cases} \tag{3.2}$$

The *a priori* estimated error in Theorem 2 is useful together with the following theorem which obtained by applying formula (3.2) inductively.

THEOREM 4. *If we subdivide a function* b^n *of degree* n k*-fold at the equidistant parameter values* $t_i = i/(k+1)(i = 1, \dots, k)$ *then the* n^{th} *forward difference* $(-1)^n V_n^{(k)} = (-1)^n \sum_{j=0}^{n} (-1)^j \binom{n}{j} c_j^{(k)}$

of the coefficients $c_i^{(k)} (i = 0, \dots, n)$ *in each of the* $k+1$ *arising segments is* $1/(k+1)^n$ *times the original difference* $(-1)^n V_n = (-1)^n \sum_{j=0}^{r} (-1)^j \binom{r}{j} b_j n$ *of the function* b^n.

Therefore, to fulfill a preset tolerance ϵ we have at first to subdivide $b^n(t)$ k-fold at t_i and afterwards to degree reduce each of the $k+1$ pieces. The number k of subdivisions is immediately calculated from Theorem 2 by

$$k = \left\lfloor \left(\frac{\alpha_n |V_n|}{\epsilon} \right)^{1/n} \right\rfloor,$$

where $\lfloor \cdot \rfloor$ denotes the floor function. Obviously, $k = 0$ means that no subdivision is necessary to fulfill the error tolerance.

Finishing this section, we briefly return to the original Problem 1 namely the reduction of the polynomial degree from n to m. A solution can be obtained by successively applying the described methods by reducing the degree by one in each step. Moreover, an upper bound of the pointwise error $d(b^n, q^m)$ can be obtained by simply adding up all the errors occurring in each step. Further, if again a tolerance ϵ should be fulfilled one have to satisfy the tolerances $\epsilon/(n-m)$ in each reduction step. This principle of stepwise degree reduction is also used in [6] and [13].

REFERENCES

1. C.S. Petersen, Adaptive contouring of three-dimensional surfaces, *Comput. Aided Geometric Des.* **1**, 61–72, (1984).
2. L. Dannenberg and H. Nowacki, Approximate conversion of surface representations with polynomial bases, *Comput. Aided Geometric Des.* **2**, 123–131, (1985).
3. J. Hoscheck, Approximate conversion of spline curves, *Comput. Aided Geometric Des.* **4**, 59–66, (1987).
4. R.G. Hersch, Font rasterization: The state of the art, Tutorial paper of Lab. de Systèmes Périphériques, (1991).
5. T.W. Sederberg and M. Kakimoto, Approximating rational curves using polynomial curves, In *NURBS for Curve and Surface Design*, (Edited by G. Farin), SIAM, Philadelphia, PA, (1991).
6. M.A. Watkins and J.A. Worsey, Degree reduction of Bézier curves, *Comput. Aided Des.* **20**, 398–405, (1988).
7. M. Eck, Degree reduction of Bézier curves, *Comput. Aided Geometric Des.* **10**, 237–251, (1993).
8. W.L.F. Degen, Best approximation of parametric curve by splines, In *Mathematical Methods in Computer Aided Design II*, (Edited by T. Lyche and L.L. Schumaker), Academic Press, New York, (1992).
9. E. Eisele, Chebyshev approximation of planar curves by splines, *J. Approx. Theory*, (1992) (to appear).
10. G. Farin, Algorithms for rational Bézier curves, *Comput. Aided Des.* **15**, 73–77, (1983).
11. A.R. Forrest, Interactive interpolation and approximation by Bézier polynomials, *Comput. Aided Des.* **22**, 527–537, (1990).
12. G. Farin, *Curves and Surfaces for Computer Aided Geometric Design*, Academic Press, New York, (1988).
13. M.A. Lachance, Chebyshev economization for parametric surfaces, *Comput. Aided Geometric Des.* **5**, 195–208, (1988).